Calculation of Covariant Dispersion Equations for Moving Plasmas

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The four-dimensional treatment of wave propagation in an homogeneous anisotropic plasma leads to a four-dimensional algebraic wave equation, where the solubility condition (dispersion equation) is given by the vanishing of a 3×3 subdeterminant. The covariant representation of such a defect algebraic system of equations is treated generally for N dimensions and its solubility condition is given by the vanishing of a determinant (having the same rank as the tensor operator of the system), whose elements are covariant themselves. The dispersion equation is explicitely given for a gyrotropic plasma, where the four-dimensional conductivity tensor can be represented by four-dimensional projectors.

1. Introduction

In the literature dispersion equations for moving media, e.g. for moving plasmas, have been calculated mostly in three-dimensional forms and for special coordinate systems, as for example [1]. Furthermore only terms of first order in v/c [2, 3] have been taken into account. Seldom dispersion relations have been calculated in four-dimensional form, as Heintzmann and Schrüfer [4] did, treating the special case of a cold electron plasma.

In the following we will establish a Lorentz covariant and coordinate invariant dispersion relation for arbitrary homogeneous anisotropic plasmas, derived from the covariant wave equation for the four-dimensional vector potential. Usually a dispersion relation is given by the solubility condition (vanishing of the determinant of the coefficient matrix) of the algebraic (i.e. Fourier transformed) wave equation. As the determinant of the 4×4 coefficient matrix vanishes identically, the solubility condition is given by the vanishing of any 3×3 subdeterminant, which are equivalent. In order to express the vanishing of such a subdeterminant in a covariant form, we follow the procedure in [4, 5]. Here the unknowns and the tensor operator are represented by two different bases. Thus the 4×4 coefficient matrix is reduced to a 3×3 matrix, thereby conserving the covariance.

As we have used two different bases the determinant of the coefficient matrix and therefore the dispersion relation become in general dependent on the two chosen bases. In the appendix it is shown what conditions, besides those already discussed in [5], the two bases have to satisfy to ensure the determinant to be independent of them.

The so obtained covariant and coordinate invariant expression of the dispersion equation for arbitrary anisotropic plasmas is then specialized for a gyrotropic plasma, which leads to a simple expression and contains the results of Heintzmann and Schrüfer [4] for a gyrotropic cold electron plasma as well as the results of Lee and Lo [1] for an uniaxial plasma.

Although the method for the calculation of covariant dispersion relations has been applied for an anisotropic plasma, the application is not restricted to a plasma and may be used for arbitrary homogeneous, bianisotropic media.

2. Covariant Algebraic Wave Equation for the Vector Potential

Maxwell's equations in covariant form are

$$\partial_\gamma G^{\nu\gamma} = j^\nu\,, \quad \partial_\gamma F_{\mu\nu} + \partial_\nu F_{\gamma\mu} + \partial_\mu F_{\nu\gamma} = 0 \quad (1)$$

with

$$G^{\nu\gamma}\,(G^{0k} = c\,D^k, G^{lk} = \varepsilon^{lkm}\,H_m) \quad \text{and}$$
$$F_{\nu\mu}\,(F_{l0} = E_l, F_{lk} = \varepsilon_{lkm}\,c\,B^m)$$

as excitation and field tensor; $j^\nu = (\varrho c, j^1, j^2, j^3)$ the four-dimensional current density vector. The greek indices are running, unless otherwise stated, from 0 to 3.

Covariant expressions for the constitutive relations are [6, 7]

$$G^{\nu\gamma} = \tfrac{1}{2}\,\lambda^{\nu\gamma}{}_{\lambda\varkappa}\,F^{\lambda\varkappa}\,,$$
$$(\delta^\nu_\mu + u^\nu\,u_\mu)\,j^\mu = \tfrac{1}{2}\,\sigma^\nu{}_{\lambda\varkappa}\,F^{\lambda\varkappa} \quad (2)$$

with u_ν the normalized four-velocity ($u_\nu u^\nu = -1$). The material tensors of order three and four,

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respectively, can be represented by four-dimensional material tensors of order two, which are closely related to the usual three-dimensional electric and magnetic permittivity and to the conductivity. In the case of a plasma anisotropic in the comoving frame, to which we restrict ourselves in order to avoid too complicated expressions, one gets [6, 7]

$$\lambda^{\nu\gamma}{}_{\lambda\varkappa} = (\varepsilon_0/\mu_0)^{1/2}(\delta^\nu_\lambda \delta^\gamma_\varkappa - \delta^\nu_\varkappa \delta^\gamma_\lambda)$$
$$\sigma^\nu{}_{\lambda\varkappa} = \sigma^\nu{}_\lambda u_\varkappa - \sigma^\nu{}_\varkappa u_\lambda \tag{3}$$

with δ^ν_λ the four-dimensional Kronecker-Tensor.

The ansatz for the vector potential

$$F_{\lambda\varkappa} = \partial_\lambda \hat{A}_\varkappa - \partial_\varkappa \hat{A}_\lambda\,,$$

and the plane wave ansatz (as we consider an homogeneous plasma)

$$\hat{A}_\lambda = A_\lambda \exp(i\, k_\alpha x^\alpha)$$

lead to an algebraic wave equation for the four-dimensional vector potential:

$$N_\nu{}^\varkappa A_\varkappa = 0 \tag{4}$$

with the wave operator

$$N_\nu{}^\varkappa := (\delta^\mu_\nu + u_\nu u^\mu)(k_\mu k^\varkappa - k_\gamma k^\gamma \delta^\varkappa_\mu) + i\sqrt{\mu_0/\varepsilon_0}\, \sigma_\nu{}^\mu (k_\mu u^\varkappa - k_\gamma u^\gamma \delta^\varkappa_\mu)\,. \tag{5}$$

From the properties of the three partially normalized projectors occurring in (5) and the relation $u^\nu \sigma_\nu{}^\mu \sim u^\mu$ [7], it follows that the rows and columns, respectively, of the matrix $N_\nu{}^\varkappa$ are linearly dependent, i.e.

$$u^\nu N_\nu{}^\varkappa = 0, \quad N_\nu{}^\varkappa k_\varkappa = 0\,. \tag{6}$$

Thus the determinant of $N_\nu{}^\varkappa$ vanishes identically. The solubility condition, and therefore the dispersion relation, is given by the vanishing of any 3×3 subdeterminant.

3. Covariant Calculation of the Dispersion Equation

The dispersion relation is calculated by a mathematical method described in the Appendix A. We construct two bases — according to (A.5) — which are not necessarily dual. If the base vectors of the four-dimensional flat space and their reciprocals are used, the bases are dual and the dispersion equation is independent of the choice of the bases. If one avoids the construction of the reciprocals — choosing two arbitrary bases — these bases have to satisfy condition (A.20) to ensure the independence of the dispersion equation on the choice of the bases.

Equations (6) do not only show the linear dependence of the rows and columns of $N_\nu{}^\varkappa$, but represent simultaneously two eigenvalue equations; u^ν is a left eigenvector of eigenvalue zero, $k_\varkappa$ the right eigenvector of the same eigenvalue.

The splitting of the conductivity tensor in a symmetric and skew-symmetric part

$$(\mu_0/\varepsilon_0)^{1/2}\, \sigma_\nu{}^\mu = S^\mu_\nu + T_\nu{}^\mu \tag{7}$$

with

$$T_\nu{}^\mu = -T^\mu{}_\nu \tag{8}$$

enables to construct, according to (A.5), two bases, consisting of $k_\varkappa$, u^ν and the contractions of the skew-symmetric tensor $T_\nu{}^\mu$ with k_μ in the form

$$\text{base I:}\quad k_\varkappa,\ T_\varkappa{}^\gamma k_\gamma/N_1,\ u_\varkappa/N_2,\ \mathscr{T}_\varkappa{}^\gamma k_\gamma/N_3 \tag{9}$$

with

$$N_1 := k^\delta T^\beta{}_\delta T_\beta{}^\alpha k_\alpha, \quad N_2 := k_\gamma u^\gamma N_1,$$
$$N_3 := k_\gamma u^\gamma u^\beta \mathscr{T}_\beta{}^\alpha k_\alpha\,, \tag{10}$$

and

$$\text{base II:}\quad u^\nu,\ T^\nu{}_\gamma k^\gamma,\ T^\nu{}_\gamma T^\gamma{}_\beta k^\beta,\ k^\nu \tag{11}$$

where the linear independence of the vectors is easily proved in the comoving frame and $\mathscr{T}^{\nu\mu}$ is the dual tensor of $T_{\lambda\varkappa}$, viz.

$$\mathscr{T}^{\nu\mu} := \tfrac{1}{2}\, \varepsilon^{\nu\mu\lambda\varkappa} T_{\lambda\varkappa} \tag{12}$$

with $\varepsilon^{\nu\mu\lambda\varkappa}$ the Levi-Civita symbol.

The same procedure as in the Appendix A (representation of $A_\varkappa$ by base I and subsequent left multiplication with the vectors of base II) leads to (A.11)

$$\det q^j{}_l = 0 \tag{13}$$

with the elements of the matrix $q^j{}_l$ given by

$$N_1 q^1{}_1 = k_\gamma k^\gamma N_1 + i\, k^\delta T^\nu{}_\delta S^\mu_\nu T_\mu{}^\gamma k_\gamma k_\beta u^\beta\,,$$
$$i N_2 q^1{}_2 = T^\nu{}_\delta k^\delta S^\mu_\nu k_\mu + N_1\,,$$
$$-i N_3 k_\alpha u^\alpha q^1{}_3 = T^\nu{}_\delta k^\delta S^\mu_\nu k_\mu N_3 - (k_\alpha u^\alpha)^2 T^\nu{}_\delta k^\delta S^\mu_\nu \mathscr{T}_\mu{}^\gamma k_\gamma + N_1 N_3\,, \tag{14}$$
$$-i N_1 q^2{}_1 = T^\nu{}_\delta T^\delta{}_\alpha k^\alpha S^\mu_\nu T_\mu{}^\gamma k_\gamma k_\beta u^\beta - \tfrac{1}{2} k_\beta u^\beta T_{\lambda\varkappa} T^{\lambda\varkappa} N_1\,,$$
$$N_2 q^2{}_2 = N_2 - i\, k_\nu T^\nu{}_\gamma T^\gamma{}_\beta S^\beta_\mu k^\mu\,,$$
$$-i N_3 k_\alpha u^\alpha q^2{}_3 = k^\nu T_\nu{}^\gamma T_\gamma{}^\beta S^\mu_\beta k_\mu N_3 - (k_\alpha u^\alpha)^2 k^\nu T_\nu{}^\gamma T_\gamma{}^\beta S^\mu_\beta \mathscr{T}_\mu{}^\varrho k_\varrho\,,$$

$$-i N_1 q^3{}_1 = k_\alpha u^\alpha k^\nu S^\mu_\nu T_\mu{}^\gamma k_\gamma + N_2 ,$$
$$N_2 q^3{}_2 = k_\alpha k^\alpha k_\beta u^\beta + (k_\alpha u^\alpha)^3 - i k^\nu S^\mu_\nu k_\mu - i k_\alpha u^\alpha k^\nu S^\mu_\nu u_\mu ,$$
$$N_3 k_\alpha u^\alpha q^3{}_3 = - k_\alpha k^\alpha k_\beta u^\beta N_3 + i k^\nu S^\mu_\nu k_\mu N_3 - i (k_\alpha u^\alpha)^2 k^\nu S^\mu_\nu \mathcal{T}_\mu{}^\gamma k_\gamma .$$

As each of the elements of the matrix $q^j{}_l$ is given in a covariant form, (13) itself is covariant and represents the covariant dispersion equation for anisotropic plasmas.

Although the expressions (14) depend on the two bases (9), (11), the determinant itself is independent of them, if, as shown in the Appendix A, the vectors

$$T_\varkappa{}^\gamma k_\gamma / N_1 , \quad (\delta^\gamma_\varkappa k_\beta k^\beta - k_\varkappa k^\gamma) u_\gamma / N_2 k_\lambda k^\lambda , \quad \mathcal{T}_\varkappa{}^\gamma k_\gamma / N_3 ;$$
$$T^\nu{}_\delta k^\delta, \quad T^\nu{}_\delta T^\delta{}_\alpha k^\alpha, \quad (\delta^\nu_\gamma + u^\nu u_\gamma) k^\gamma \tag{15}$$

fulfill condition (A.22). The vectors (15) are the projections of the last three vectors of both bases (9) and (11) into the subspaces orthogonal to the right and left eigenvectors of eigenvalue zero, $k_\varkappa$ and u^ν, respectively. They correspond to $a^{l'}_\varkappa$ and $b^\nu_{j'}$ in the Appendix A.

With the second of the relations

$$S^\nu_\lambda u_\nu \sim u_\lambda , \quad S^\nu_\lambda u^\lambda \sim u^\nu ,$$
$$T^\nu{}_\lambda u^\lambda = 0 = T^\nu{}_\lambda u_\nu , \tag{16}$$

which follow from $\sigma_\nu{}^\lambda u_\lambda \sim u_\nu$ [7] and can most conveniently be verified in the comoving frame, and with the relations between a skew-symmetric tensor (satisfying relations (16)) and its dual [8, 9].

$$T^\alpha{}_\beta \mathcal{T}^\beta{}_\gamma = 0 ,$$
$$\mathcal{T}^\alpha{}_\beta \mathcal{T}^\beta{}_\gamma = - T^\alpha{}_\beta T^\beta{}_\gamma - \tfrac{1}{2} \delta^\alpha_\gamma T^{\lambda\varkappa} T_{\lambda\varkappa} , \tag{17}$$

it can be shown that the vectors (15) satisfy (A.22).

4. Specialization to Gyrotropic Plasmas

In a gyrotropic plasma the anisotropy is caused by an outer magnetic field, represented by an axial vector B^l. Therefore the plasma is represented by an axial conductivity tensor, which is built up by the magnetic field vector. This axial conductivity tensor can be represented by projectors ${}_{M'}P_\nu{}^\mu$ (B.4) according to Eq. (B.9) in Appendix B, or using the tensors $\mathrm{Re}\, {}_1P_\nu{}^\mu$, $\mathrm{Im}\, {}_1P_\nu{}^\mu$ (B.10) instead of ${}_1P_\nu{}^\mu$ and ${}_{-1}P_\nu{}^\mu$ by (cf. (B.11))

$$\sqrt{\mu_0/\varepsilon_0}\, \sigma_\nu{}^\mu = \sigma_0\, {}_0P_\nu{}^\mu + \sigma_+ 2\, \mathrm{Re}\, {}_1P_\nu{}^\mu + i \sigma_- 2\, \mathrm{Im}\, {}_1P_\nu{}^\mu + \sigma_2\, {}_2P_\nu{}^\mu . \tag{18}$$

The eigenvalue $(\varepsilon_0/\mu_0)^{1/2} \sigma_2$ is the pure time component of $\sigma_\nu{}^\mu$ in the comoving frame and is therefore arbitrary [7]. The symmetric and skew-symmetric parts of $\sigma_\nu{}^\mu$ are given with (B.10), (B.4) as

$$S^\mu_\nu = \sigma_0\, {}_0P_\nu{}^\mu + \sigma_+ 2\, \mathrm{Re}\, {}_1P_\nu{}^\mu + \sigma_2\, {}_2P_\nu{}^\mu = \sigma_0 b_\nu b^\mu + \sigma_+ (\delta^\mu_\nu - b_\nu b^\mu + u_\nu u^\mu) - \sigma_2 u_\nu u^\mu , \tag{19}$$
$$T_\nu{}^\mu = i \sigma_- 2\, \mathrm{Im}\, {}_1P_\nu{}^\mu = i \sigma_- \varepsilon_{\gamma\nu}{}^{\delta\mu} b_\delta u^\gamma$$

with b^ν as a unit vector of the magnetic four-field $B^\nu =: \frac{1}{2} \varepsilon^{\nu\mu\lambda\varkappa} F_{\lambda\varkappa} u_\mu / c$ [7].

These expressions reduce the elements (14) of the matrix $q^j{}_l$ to

$$q^1{}_1 = k_\gamma k^\gamma + i \sigma_+ k_\gamma u^\gamma , \quad q^1{}_2 = - i / k_\gamma u^\gamma ,$$
$$q^1{}_3 = i \sigma_-{}^2 [k_\gamma k^\gamma + (k_\gamma u^\gamma)^2 - (k_\gamma b^\gamma)^2] / k_\beta u^\beta ,$$
$$q^2{}_1 = i k_\gamma u^\gamma \sigma_-{}^2 , \quad q^2{}_2 = (k_\gamma u^\gamma - i \sigma_+) / k_\beta u^\beta ,$$
$$q^2{}_3 = i \sigma_+ \sigma_-{}^2 [k_\gamma k^\gamma + (k_\gamma u^\gamma)^2 - (k_\gamma b^\gamma)^2] / k_\beta u^\beta ,$$
$$q^3{}_1 = i k_\gamma u^\gamma , \tag{20}$$
$$q^3{}_2 k_\gamma u^\gamma \sigma_-{}^2 [k_\beta k^\beta + (k_\beta u^\beta)^2 - (k_\beta b^\beta)^2] = (k_\gamma u^\gamma - i \sigma_+) [k_\beta k^\beta + (k_\beta u^\beta)^2] - i (\sigma_0 - \sigma_+) (k_\beta b^\beta)^2 ,$$
$$q^3{}_3 k_\gamma u^\gamma = - k_\gamma k^\gamma (k_\beta u^\beta - i \sigma_+) + i (\sigma_0 - \sigma_+) [(k_\gamma b^\gamma)^2 - (k_\gamma u^\gamma)^2] ,$$

where the arbitrary σ_2 has dropped out and the following relations have been used:

$$T_\nu{}^\mu S^\lambda_\mu = \sigma_+ T_\nu{}^\lambda , \quad \mathcal{T}_\nu{}^\mu = \sigma_- (u_\nu b^\mu - b_\nu u^\mu) ,$$
$$k^\alpha S^\varkappa_\alpha k_\varkappa = (\sigma_0 - \sigma_+) (k_\gamma b^\gamma)^2 + \sigma_+ k_\gamma k^\gamma - (\sigma_2 - \sigma_+) (k_\gamma u^\gamma)^2 ,$$
$$k^\alpha S^\varkappa_\alpha u_\varkappa = \sigma_2 k_\gamma u^\gamma , \tag{21}$$
$$k^\alpha S^\varkappa_\alpha \mathcal{T}_\varkappa{}^\gamma k_\gamma = - \sigma_- k_\alpha b^\alpha k_\gamma u^\gamma (\sigma_0 - \sigma_2) ,$$
$$u_\varkappa \mathcal{T}^\varkappa{}_\gamma k^\gamma = - \sigma_- k_\gamma b^\gamma , \quad T_{\lambda\varkappa} T^{\lambda\varkappa} = - 2 \sigma_-{}^2 ,$$
$$k^\nu T_\nu{}^\varkappa T_\varkappa{}^\mu k_\mu = \sigma_-{}^2 [k_\gamma k^\gamma + (k_\gamma u^\gamma)^2 - (k_\gamma b^\gamma)^2] .$$

The calculation of the determinant (13) leads — after some manipulations — to the explicit form of the covariant dispersion equation in gyrotropic plasmas:

$$0 = \det q^j{}_l = [k_\gamma k^\gamma + i \sigma_0 k_\gamma u^\gamma] \cdot \{[k_\beta k^\beta + i \sigma_+ k_\beta u^\beta] [i \sigma_+ - k_\alpha u^\alpha] + \sigma_-{}^2 k_\beta u^\beta\} + \{i (\sigma_0 - \sigma_+) [k_\gamma k^\gamma + i \sigma_+ k_\gamma u^\gamma] - \sigma_-{}^2 k_\gamma u^\gamma\} \cdot (k_\beta b^\beta)^2 , \tag{22}$$

where the properties of the plasma are characterized by the eigenvalues

$$\sqrt{\varepsilon_0/\mu_0}\, \sigma_0, \quad \sqrt{\varepsilon_0/\mu_0}\, \sigma_{\pm 1} = \sqrt{\varepsilon_0/\mu_0}\, (\sigma_+ \pm \sigma_-)$$

of the conductivity tensor $\sigma_\nu{}^\mu$. With the relations (B.15) one can formulate the dispersion equation in terms of the inverse eigenvalues $f_{M'}$, as

$$\begin{aligned}&(i\,k_\gamma k^\gamma f_0 - k_\gamma u^\gamma)\\&\cdot[(1+i\,k_\beta u^\beta f_+)(i\,k_\alpha k^\alpha f_+ - k_\alpha u^\alpha) + k_\beta k^\beta k_\alpha u^\alpha f_-{}^2]\\&+[k_\gamma k^\gamma f_-{}^2 - i(i\,k_\gamma k^\gamma f_+ - k_\gamma u^\gamma)(f_0 - f_+)](k_\alpha b^\alpha)^2\\&=0\,. \qquad (23)\end{aligned}$$

For a cold collisionless plasma of density n_e the reciprocal conductivity tensor in the comoving frame is given by [2]

$$\begin{aligned}&\sqrt{\varepsilon_0/\mu_0}\,(\sigma^{-1})'^l{}_k\\&=(c/\omega_P'^2)(-i\,\omega'\,\delta^l_k - \omega_B'\,\varepsilon^l{}_{km}\,\hat{B}'^m)\,,\end{aligned}$$

$$\omega_P'^2 := \frac{q_e^2\,n_e'}{m_e'\,\varepsilon_0} = \frac{q_e^2\,n_e}{m_e\,\varepsilon_0} = \omega_P^2\,,\quad \omega_B' := \frac{q_e\,B'}{m_e'}\,,$$

which leads to the following expressions for f_+', f_0', f_-':

$$f_0' = f_+' = -\frac{i\,\omega'\,c}{\omega_P'^2} = \frac{i\,k_\gamma u^\gamma c^2}{\omega_P^2} = f_+ = f_0\,,$$

$$f_-' = \frac{i\,c\,\omega_B'}{\omega_P^2} \qquad (24)$$

with $\omega_P/2\pi$ the Lorentz invariant plasma frequency and $\omega_B/2\pi$ the gyrofrequency.

Inserting f_+ and f_0 in (23) and including f_- into $T_\nu{}^\mu$ according to the second equation of (19), one gets with relations (21)

$$\begin{aligned}&\left\{(k_\gamma u^\gamma)^2\left[k_\alpha k^\alpha + \frac{\omega_P^2}{c^2}\right]^2 - \frac{1}{2}\frac{\omega_P^4}{c^4}\,T^{\alpha\beta}\,T_{\alpha\beta}(k_\delta k^\delta)^2\right\}\\&\cdot\left[(k_\nu u^\nu)^2 - \frac{\omega_P^2}{c^2}\right] + \frac{\omega_P^6}{c^6}\,k^\alpha\,T_{\alpha\beta}\,T^{\beta\delta}\,k_\delta\,k_\gamma k^\gamma = 0 \qquad (25)\end{aligned}$$

as the covariant dispersion equation for a lossless cold electron plasma. With $(\omega_P/c)^2\,T_\alpha{}^\beta \equiv F_\alpha{}^\beta$ this is the dispersion equation of Heintzmann and Schrüfer [4].

When the magnetic field becomes very large the conductivity tensor becomes uniaxial. In this case the gyrofrequency $\omega_B/2\pi$ increases infinitely and consequently f_- (24), hence the quantities $\sigma_\pm$, given by (24) together with (B.16), vanish. The dispersion equation (22) then simplifies to the factorized form

$$\begin{aligned}&k_\gamma k^\gamma[k_\alpha k^\alpha k_\beta u^\beta + i\,\sigma_0(k_\alpha u^\alpha)^2\\&\qquad - i\,\sigma_0(k_\alpha b^\alpha)^2] = 0\,. \qquad (26)\end{aligned}$$

The vanishing of the expression between the brackets leads to the dispersion relation for the extraordinary mode, while $k_\gamma k^\gamma = 0$ is the dispersion relation for the ordinary mode (a vacuum mode), not influenced by the convection. Equation (26) includes the expressions given by Lee and Lo [1] for the special case $\boldsymbol{v} \parallel \boldsymbol{B} \parallel \boldsymbol{e}_2$.

For a plasma isotropic in the comoving frame ($B'^\nu = 0 = f_-' = \sigma_-'$) there holds

$$\sigma_+ = \frac{1}{f_+} = \frac{\omega_P^2}{i\,k_\gamma u^\gamma c^2} = \frac{1}{f_0} = \sigma_0$$

and the dispersion equation (23) becomes

$$\left[k_\gamma k^\gamma + \frac{\omega_P^2}{c^2}\right]^2 [\omega_P^2 - (k_\beta u^\beta)^2 c^2] = 0\,. \qquad (27)$$

The vanishing of the first bracket leads to the dispersion relation for the electromagnetic waves, which is not influenced by the convection as long as no collisions are taken into account, while the second term is the covariant representation of the relation $\omega' = \pm\,\omega_P'$ i.e. the expression for the plasma oscillations in the comoving frame. In the observer's frame they become

$$(\omega - k_l v^l)\,/\,(1 - v^2/c^2)^{1/2} = \pm\,\omega_P\,.$$

Concluding Remarks

The method for the calculation of a covariant dispersion relation as described in this paper is not restricted to plasmas. It may be applied wherever one establishes a covariant dispersion relation from an algebraic wave equation, where the dispersion equation (solubility condition) is given by the vanishing of a subdeterminant of the tensor operator of the wave equation.

The method, viz. representation of the unknown and the tensor operator by two different bases, makes in general the dispersion equation dependent of the choice of the bases. In the Appendix A we discussed various constraints, besides the trivial case of duality, for the two bases, which ensure the independence of the dispersion equation of the choice of the bases and avoid the construction of the dual base.

Applying this method we established a covariant dispersion equation for an homogeneous anisotropic plasma. For gyrotropic media the three-dimensional material tensors in the comoving frame can be represented by projectors [10]. We generalized this representation to four dimensions (Appendix B) and calculated the covariant dispersion relation for a gyrotropic plasma. The obtained expressions coincide for special cases with existing results [1], [4], [11].

Appendix

A) Covariant Representation of the Solubility Condition of an Homogeneous System of Equations

We consider an homogeneous system of $N+1$ equations with $N+1$ unknowns

$$N_\alpha{}^\beta A_\beta = 0\,, \tag{A.1}$$

where throughout the Appendix A the greek indices are running from 0 to N, the capital latin indices from 0 to K and the lower case latin indices from $K+1$ to N. If the rank of the matrix $N_\alpha{}^\beta$ is $N+1$, the solubility condition is simply given by $\det N_\alpha{}^\beta = 0$. We exclude this trivial case and consider in the following

$$\operatorname{rank} N_\alpha{}^\beta = N - K, \quad K \geqq 0\,.$$

Thereby the number of the independent unknowns shall be equal to the number of the independent equations; the system is uniquely determined. Thus the number of the independent rows and columns of $N_\alpha{}^\beta$ is equal and there exists only one significant $(N-K)\times(N-K)$ subdeterminant, whose vanishing provides the solubility condition.

For such a matrix $N_\alpha{}^\beta$ the following eigenvalue equations exist:

$$N_\alpha{}^\beta x_\beta^{\nu'} = \lambda^{(\nu')} x_\alpha^{\nu'}\,, \quad y_{\mu'}^\alpha N_\alpha{}^\beta = \varrho^{(\mu')} y_{\mu'}^\beta\,, \tag{A.2}$$

where

$$\begin{aligned} &\lambda^{(A')} = 0, \quad \varrho^{(A')} = 0\,, \quad A' = 0,1,\dots,K; \\ &\lambda^{(l')} \neq 0, \quad \varrho^{(l')} \neq 0\,, \quad l' = K+1,\dots,N \end{aligned} \tag{A.3}$$

with $\lambda^{(\nu')}$ ($\varrho^{(\mu')}$) as right (left) eigenvalues and $x_\alpha^{\nu'}$ ($y_{\mu'}^\beta$) as right (left) eigenvectors. Throughout the appendices the Einstein summation convention is only valid for unprimed indices, while the primed ones are merely labels. As the set of left and right eigenvalues is equal, one can arrange them appropriately ($\lambda^{(\nu')} = \varrho^{(\nu')}$) and gets with

$$y_{\mu'}^\alpha N_\alpha{}^\beta x_\beta^{\nu'} = \lambda^{(\nu')} y_{\mu'}^\alpha x_\alpha^{\nu'} = \varrho^{(\mu')} y_{\mu'}^\beta x_\beta^{\nu'}$$

the biorthogonality relations

$$\begin{aligned} y_{l'}^\alpha x_\alpha^{m'} \sim \delta_{l'}^{m'}, \quad y_{l'}^\alpha x_\alpha^{A'} &\sim \delta_{l'}^{A'} = 0, \\ y_{B'}^\alpha x_\alpha^{m'} &\sim \delta_{B'}^{m'} = 0\,, \\ B', A' = 0,1,\dots,K, \quad l', m' &= K+1,\dots,N\,. \end{aligned} \tag{A.4}$$

One cannot make a statement about $x_\alpha^{A'} y_{B'}^\alpha$ since both eigenvalues $\lambda^{(A')}$ and $\varrho^{(B')} = \lambda^{(B')}$ are zero (A.3).

With the $2\times(K+1)$ right and left eigenvectors $x_\alpha^{A'}, y_{B'}^\alpha$ of the eigenvalue zero and arbitrary $2\times(N-K)$ vectors $c_\beta^{k'}, d_{j'}^\alpha$ one can build up two bases

$$\begin{aligned} &\text{base I:} \quad x_\beta^{A'}, c_\beta^{k'} \\ &\text{base II:} \quad y_{B'}^\alpha, d_{j'}^\alpha\,. \end{aligned} \tag{A.5}$$

The vectors $c_\beta^{k'}$ and $d_{j'}^\alpha$, respectively, must not lie in the subspaces of the $x_\beta^{m'}$ and $y_{l'}^\alpha$, respectively, but have to be mutually independent. They may be represented by

$$\begin{aligned} c_\beta^{k'} &= a_\beta^{k'} + \sum_{A'=0}^{K} C^{k'}{}_{A'} x_\beta^{A'}\,, \\ d_{j'}^\alpha &= b_{j'}^\alpha + \sum_{B'=0}^{K} D_{j'}{}^{B'} y_{B'}^\alpha \end{aligned} \tag{A.6}$$

with $a_\beta^{k'}$ ($b_{j'}^\alpha$) as the projections of $c_\beta^{k'}$ ($d_{j'}^\alpha$) into the $(N-K)$-dimensional subspace of the eigenvectors with eigenvalue unequal to zero. According to (A.4) these projections obey the biorthogonality relations

$$x_\alpha^{A'} b_{j'}^\alpha = 0\,, \quad y_{B'}^\alpha a_\alpha^{k'} = 0\,. \tag{A.7}$$

Now the $(N+1)$-dimensional unknown vector A_β (A.1) is represented by base I as:

$$A_\beta = \sum_{A'=0}^{K} C_{A'} x_\beta^{A'} + \sum_{k'=K+1}^{N} C_{k'} c_\beta^{k'}\,.$$

Substitution into (A.1) leads according to (A.2) to

$$\sum_{k'=K+1}^{N} N_\alpha{}^\beta c_\beta^{k'} C_{k'} = 0\,,$$

where now the number of unknowns $C_{k'}$ is reduced to $N-K$. Subsequent multiplications with the vectors $y_{B'}^\alpha$ and $d_{j'}^\alpha$ of base II eliminate $K+1$ redundant equations and reduce the $(N+1)\times(N+1)$ system (A.1) to the $(N-K)\times(N-K)$ system

$$\sum_{k'=K+1}^{N} q_{j'}{}^{k'} C_{k'} = 0 \tag{A.8}$$

with the coefficients

$$q_{j'}{}^{k'} := d_{j'}^\alpha N_\alpha{}^\beta c_\beta^{k'} \tag{A.9}$$

or, according to (A.6) and the eigenvalue equations (A.2),

$$q_{j'}{}^{k'} = b_{j'}^\alpha N_\alpha{}^\beta a_\beta^{k'}\,. \tag{A.10}$$

The solubility condition of the $(N-K)\times(N-K)$ system (A.8) is given by

$$\det q_{j'}{}^{k'} = 0\,. \tag{A.11}$$

As we have used two, in general different, bases and as the elements of the matrix $q_{j'}{}^{k'}$ depend on the two bases, the determinant (A.11) itself depends on the choice of the bases. Two other bases

base $\hat{\mathrm{I}}$: $\quad x_\beta^{A''},\ \hat{c}_\beta^{s''}$

base $\hat{\mathrm{II}}$: $\quad y^\alpha_{B''},\ \hat{d}^\alpha_{t''}$

would lead to

$$\hat{q}_{t''}{}^{s''} := \hat{b}^\alpha_{t''}\, N_\alpha{}^\beta\, \hat{a}_\beta^{s''}, \tag{A.12}$$

where now $\hat{b}^\alpha_{t''}$ ($\hat{a}_\beta^{s''}$) are the projections of $\hat{d}^\alpha_{t''}$ ($\hat{c}_\beta^{s''}$) into the subspace of the vectors $y^\alpha_{l'}$ ($x_\beta^{m'}$). The solubility condition is

$$\det \hat{q}_{t''}{}^{s''} = 0\,. \tag{A.13}$$

The first $K+1$ vectors of the bases I and $\hat{\mathrm{I}}$ "transform" as

$$x_\beta^{A''} = \delta^{A''}_{A'}\, x_\beta^{A'},$$

as the eigenvectors of the eigenvalue zero are not transformed. The vectors $\hat{c}_\beta^{s''}$ of base $\hat{\mathrm{I}}$ are expressed by the vectors $x_\beta^{A'}$, $c_\beta^{k'}$ of base I (A.5) as

$$\hat{c}_\beta^{s''} = \sum_{A'=0}^{K} A^{s''}{}_{A'}\, x_\beta^{A'} + \sum_{k'=K+1}^{N} A^{s''}{}_{k'}\, c_\beta^{k'}\,.$$

Relation (A.6) and the analogous relation for $\hat{c}_\beta^{s''}$ lead to

$$\begin{aligned}\hat{c}_\beta^{s''} &= \hat{a}_\beta^{s''} + \sum C^{s''}{}_{A'}\, x_\beta^{A'}\\ &= \sum_{A'=0}^{K} A^{s''}{}_{A'}\, x_\beta^{A'} + \sum_{k'=K+1}^{N} A^{s''}{}_{k'}\, a_\beta^{k'}\\ &\quad + \sum_{k'=K+1}^{N} A^{s''}{}_{k'} \sum_{A'=0}^{K} C^{k'}{}_{A'}\, x_\beta^{A'}\,.\end{aligned}$$

Contraction with a set of eigenvectors $y^\beta_{A''}$ made biorthogonal with the set $x_\beta^{A'}$, i.e. $y^\beta_{A''}\, x_\beta^{A'} = \delta^{A'}_{A''}$ yields

$$C^{s''}{}_{A'} = A^{s''}{}_{A'} + \sum_{k'=K+1}^{N} A^{s''}{}_{k'}\, C^{k'}{}_{A'}\,. \tag{A.14a}$$

Contraction with any of the vectors $b^\beta_{t''}$ yields

$$\hat{a}_\beta^{s''} - \sum_{k'=K+1}^{N} A^{s''}{}_{k'}\, a_\beta^{k'} = 0\,, \tag{A.14b}$$

where the above vector is identical with the null-vector, because it lies in the subspace of the $x_\beta^{m'}$ and therefore, according to (A.4), it cannot be orthogonal to the subspace of the $b^\alpha_{j'}$ (equal to the subspace of the $y^\alpha_{l'}$). Equation (A.14b) shows, that the transformation within the subspaces of the eigenvectors of eigenvalue unequal to zero is independent of the transformation within the subspaces of the eigenvectors of eigenvalue zero. According to (A.14b) $\hat{b}^\alpha_{t''}$, $b^\alpha_{t'}$ transform as

$$\hat{b}^\alpha_{t''} = \sum_{j'=K+1}^{N} B_{t''}{}^{j'}\, b^\alpha_{j'}\,. \tag{A.15}$$

The transformations (A.14b), (A.15) lead to the following transformation of the matrices (A.9), (A.12):

$$\begin{aligned}\hat{q}_{t''}{}^{s''} &= \hat{b}^\alpha_{t''}\, N_\alpha{}^\beta\, \hat{a}_\beta^{s''}\\ &= \sum_{k'=K+1}^{N}\ \sum_{j'=K+1}^{N} B_{t''}{}^{j'}\, q_{j'}{}^{k'}\, A^{s''}{}_{k'}\,,\end{aligned} \tag{A.16}$$

and thus the determinant transforms as

$$\det \hat{q}_{t''}{}^{s''} = \det q_{j'}{}^{k'} \det B_{t''}{}^{j'} \det A^{s''}{}_{k'}\,. \tag{A.17}$$

The determinants of the transformation matrices $A^{s''}{}_{k'}$, $B_{t''}{}^{j'}$ can be expressed explicitly by the vectors of the bases I, $\hat{\mathrm{I}}$, II, $\hat{\mathrm{II}}$, because the inner product

$$\hat{a}_\beta^{s''}\, \hat{b}^\beta_{t''} = \sum_{k'=K+1}^{N}\ \sum_{j'=K+1}^{N} A^{s''}{}_{k'}\, a_\beta^{k'}\, b^\beta_{j'}\, B_{t''}{}^{j'}$$

yields

$$\begin{aligned}&\det A^{s''}{}_{k'} \det B_{t''}{}^{j'}\\ &\quad = \det \hat{a}_\beta^{s''}\, \hat{b}^\beta_{t''} / \det a_\beta^{k'}\, b^\beta_{j'}\,.\end{aligned} \tag{A.18}$$

Relations (A.17) and (A.18) show explicitly the dependence of the determinants (A.11), (A.13) on the chosen bases.

The described method of the calculation of the solubility condition (unknown vector and tensor operator represented in different bases) is mainly a mathematical device in order to avoid the calculation of the dual bases. For in the particular case of dual bases, i.e.

$$c_\beta^{k'}\, d^\beta_{j'} = \delta^{k'}_{j'}\,, \quad \hat{c}_\beta^{s''}\, \hat{d}^\beta_{t''} = \delta^{s''}_{t''}$$

the right hand side of Eq. (A.18) becomes unity, thus making the determinants (A.11), (A.13) independent of the bases. But because we want to avoid the construction of the dual bases, the determinants (solubility conditions) depend on the bases. Thus we have to impose a constraint on the bases I, II in order to ensure that the solubility condition is independent of the bases.

Such a constraint can be the orthogonality of the vectors $a_\beta^{k'}$, $b^\alpha_{j'}$, wich requires (A.14b), (A.15) to be orthogonal transformations ($\det A^{s''}{}_{k'} = \det B_{t''}{}^{j'} = 1$). Frequently the solubility condition represents a dispersion equation ($\omega = f(\boldsymbol{k})$). In this case it is sufficient that the vectors of the bases are independent of $\boldsymbol{k}$ and ω in order to restore, beside an arbitrary factor, the functional structure of the solubility condition.

Another possible constraint is the biorthogonality of the projections (A.6) of the vectors $c_\beta^{k'}$ ($d_{j'}^\alpha$) into the subspace of the vectors $x_\beta^{m'}$ ($y_{j'}^\alpha$), viz.

$$a_\beta^{k'}\, b_{j'}^\beta = \delta_{j'}^{k'} \tag{A.19}$$

or the weaker condition

$$a_\beta^{k'}\, b_{j'}^\beta = \delta_{j'}^{k'} \quad \text{for} \quad k' \geqq j' \text{ or } k' \leqq j' \tag{A.20}$$

or the even weaker condition

$$\det a_\beta^{k'}\, b_{j'}^\beta = 1\,, \tag{A.21}$$

which ensure the solubility condition to be independent of the chosen bases.

In the case of a four-dimensional system of Eq. (A.1) with rank $N_\alpha{}^\beta = 3$ ($N=3$, $K=0$) condition (A.20) is explicitly written as

$$\begin{aligned} a_\beta^{1'}\, b_{1'}^\beta = 1, \quad a_\beta^{1'}\, b_{2'}^\beta &= 0, \quad a_\beta^{1'}\, b_{3'}^\beta = 0,\\ a_\beta^{2'}\, b_{2'}^\beta &= 1, \quad a_\beta^{2'}\, b_{3'}^\beta = 0,\\ & \quad\;\; a_\beta^{3'}\, b_{3'}^\beta = 1. \end{aligned} \tag{A.22}$$

B) Four-dimensional Representation of Axial Tensors

A three-dimensional second order axial tensor $A^l{}_k$, which is built up solely by an axial vector $\bar{B}^l = |\boldsymbol{B}|\,\hat{B}^l$, may be represented by means of the complete set of three projectors [10]

$$\begin{aligned} {}_0\hat{P}_k^l &:= \hat{B}^l\, \hat{B}_k\,,\\ {}_{\pm 1}\hat{P}^l{}_k &:= \tfrac{1}{2}\,(\delta_k^l - \hat{B}^l\, \hat{B}_k \pm i\, \varepsilon^l{}_{mk}\, \hat{B}^m)\\ &= {}_{\mp 1}\hat{P}^{*l}{}_k \end{aligned} \tag{B.1}$$

on account of the eigenvalue equations

$$A^l{}_{k\; M'}\hat{P}^k{}_j = A_{M'\; M'}\hat{P}^l{}_j\,, \quad M' = -1, 0, +1 \tag{B.2}$$

in the diagonal form

$$A^l{}_k = A_0\, {}_0\hat{P}_k^l + A_{+1}\, {}_{+1}\hat{P}^l{}_k + A_{-1}\, {}_{-1}\hat{P}^l{}_k\,. \tag{B.3}$$

With

$$\begin{aligned} 2\,\mathrm{Re}\, {}_1\hat{P}^l{}_k &= {}_{+1}\hat{P}^l{}_k + {}_{-1}\hat{P}^l{}_k = \delta_k^l - \hat{B}^l\, \hat{B}_k\,,\\ 2i\,\mathrm{Im}\, {}_1\hat{P}^l{}_k &= {}_{+1}\hat{P}^l{}_k - {}_{-1}\hat{P}^l{}_k = i\, \varepsilon^l{}_{mk}\, \hat{B}^m\,,\\ A_\pm &:= \tfrac{1}{2}\,(A_{+1} \pm A_{-1})\,, \end{aligned}$$

the tensor $A^l{}_k$ (B.3) may be represented by [10]

$$A^l{}_k = A_0\, {}_0\hat{P}_k^l + A_+\, 2\,\mathrm{Re}\, {}_1\hat{P}^l{}_k + i\, A_-\, 2\,\mathrm{Im}\, {}_1\hat{P}^l{}_k\,.$$

For the representation of a four-dimensional second order tensor (as for example the material tensors given in [7] for gyrotropic media), whose mixed space-time components vanish in the comoving frame and whose space components are given there by an axial tensor, the following four-dimensional projectors can be used:

$$\begin{aligned} {}_{\pm 1}P_\beta^\alpha &:= \tfrac{1}{2}\,(\delta_\beta^\alpha - b^\alpha\, b_\beta + u^\alpha\, u_\beta \pm i\, \varepsilon^{\gamma\alpha}{}_{\delta\beta}\, b^\delta\, u_\gamma)\\ &= {}_{\mp 1}P^{*\alpha}{}_\beta\,, \end{aligned} \tag{B.4}$$

$${}_0P_\beta^\alpha := b^\alpha\, b_\beta\,, \quad {}_2P_\beta^\alpha := -\, u^\alpha\, u_\beta\,.$$

With $b^\alpha := B^\alpha (B_\beta B^\beta)^{-1/2}$ and the property $b_\alpha u^\alpha = 0$ they satisfy the completeness and orthogonality relations

$$\begin{aligned} {}_{-1}P^\alpha{}_\beta + {}_0P_\beta^\alpha + {}_{+1}P^\alpha{}_\beta + {}_2P_\beta^\alpha &= \delta_\beta^\alpha\,,\\ {}_{M'}P^\alpha{}_{\beta\; N'}P^\beta{}_\gamma &= {}_{M'}P^\alpha{}_\gamma\, \delta_{M'N'}\,,\\ M', N' &= -1, 0, +1, 2\,. \end{aligned} \tag{B.5}$$

In the comoving frame the time component of the four-vector b'^α vanishes (because of $b^\alpha u_\alpha = 0$) and the space components b'^l become equal to the axial vector $\hat{B}^l$ (i.e. $B'^l = \hat{B}^l\,|\boldsymbol{B}|$) [7]. Hence the projectors ${}_{\pm 1}P'^\alpha{}_\beta$, ${}_0P'^\alpha_\beta$ have only pure space components and coincide with the three-dimensional projectors (B.1), i.e.

$$\begin{aligned} {}_{M'}P'^l{}_k = {}_{M'}\hat{P}^l{}_k\,, \quad {}_{M'}P'^0{}_k = {}_{M'}P'^l{}_0 = {}_{M'}P'^0{}_0 = 0\,,\\ M' = -1, 0, +1\,. \end{aligned} \tag{B.6}$$

The projector ${}_2P_\beta^\alpha = -\,u^\alpha u_\beta$ has been added for completeness and in the comoving frame is the only one with non vanishing (0, 0)-component.

If the four-vector B^α should represent a magnetic field in the comoving frame, it can be expressed by the dual electromagnetic field tensor

$$\mathscr{F}^{\alpha\beta} := \tfrac{1}{2}\, \varepsilon^{\alpha\beta\gamma\delta}\, F_{\gamma\delta}$$

in the form [7]

$$c\, B^\alpha = \mathscr{F}^{\alpha\beta}\, u_\beta \tag{B.7a}$$

and thus satisfies the condition $B^\alpha u_\alpha = 0$ because $\mathscr{F}^{\alpha\beta}$ is skew-symmetric. If there is no electric field in the comoving frame, one can express $F_{\alpha\beta}$ by [12]

$$F_{\alpha\beta} = -\, \varepsilon_{\alpha\beta\gamma\delta}\, u^\gamma\, c\, B^\delta\,. \tag{B.7b}$$

With the eigenvalue equations

$$\begin{aligned} A^\alpha{}_{\beta\; M'}P^\beta{}_\gamma = A_{M'\; M'}P^\alpha{}_\gamma\,,\\ M' = -1, 0, +1, 2 \end{aligned} \tag{B.8}$$

one can represent $A^\alpha{}_\beta$ in the diagonal form

$$\begin{aligned} A^\alpha{}_\beta = A_{+1}\, {}_{+1}P^\alpha{}_\beta + A_0\, {}_0P_\beta^\alpha + A_{-1}\, {}_{-1}P^\alpha{}_\beta\\ + A_2\, {}_2P_\beta^\alpha\,. \end{aligned} \tag{B.9}$$

The eigenvalues $A_{M'}$ are most conveniently calculated in the comoving frame as there A_0', $A_{\pm 1}'$ are the eigenvalues of the axial tensor, forming the space components of $A'^{\alpha}{}_{\beta}$, and A'_2 is the time component.

The two tensors

$$\begin{aligned} 2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta} &= {}_{+1}P^{\alpha}{}_{\beta} + {}_{-1}P^{\alpha}{}_{\beta} \\ &= \delta^{\alpha}_{\beta} - b^{\alpha}\,b_{\beta} + u^{\alpha}\,u_{\beta}\,, \\ 2\,i\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta} &= {}_{+1}P^{\alpha}{}_{\beta} - {}_{-1}P^{\alpha}{}_{\beta} \\ &= i\,\varepsilon^{\gamma\alpha}{}_{\delta\beta}\,b^{\delta}\,u_{\gamma} \end{aligned} \tag{B.10}$$

lead to the representation of $A^{\alpha}{}_{\beta}$ in the form

$$\begin{aligned} A^{\alpha}{}_{\beta} &= A_0\,{}_0P^{\alpha}_{\beta} + A_+\,2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta} \\ &\quad + i\,A_-\,2\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta} + A_2\,{}_2P^{\alpha}_{\beta} \end{aligned} \tag{B.11}$$

with the coefficients

$$\begin{aligned} A_{\pm} &:= \tfrac{1}{2}\,(A_{+1} \pm A_{-1})\,, \\ A_{\pm 1} &= A_+ \pm A_-\,. \end{aligned} \tag{B.12}$$

The two tensors $2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta}$, $2\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta}$ (B.10) can be represented by the field tensor $F^{\alpha}{}_{\beta}$, if there is no electric field in the comoving frame. Use of the relation (B.7 b) together with the properties of the Levi-Civita symbol yields

$$\begin{aligned} 2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta} &= -\,F^{\alpha}{}_{\gamma}\,F^{\gamma}{}_{\beta}/c^2\,B_{\delta}\,B^{\delta}\,, \\ 2\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta} &= F^{\alpha}{}_{\beta}/(c^2\,B_{\delta}\,B^{\delta})^{1/2}\,. \end{aligned} \tag{B.13}$$

With ${}_0P^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + u^{\alpha}u_{\beta} - 2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta}$ ((B.4), (B.10)) the representation (B.11) becomes

$$\begin{aligned} A^{\alpha}{}_{\beta} &= (A_0 - A_+)\,F^{\alpha}{}_{\gamma}\,F^{\gamma}{}_{\beta}/c^2\,B_{\delta}\,B^{\delta} \\ &\quad + i\,A_-\,F^{\alpha}{}_{\beta}(c\,B_{\delta}\,B^{\delta})^{-1/2} \\ &\quad + (A_0 - A_2)\,u^{\alpha}\,u_{\beta}\,. \end{aligned} \tag{B.14}$$

For the special case $A_0 = A_2$ this is the representation of Derfler and O'Sullivan [11].

The diagonal representation (B.9) of $A^{\alpha}{}_{\beta}$ by the projectors ${}_{M'}P^{\alpha}{}_{\beta}$ is most convenient for inversions. The orthogonality relations (B.5) yield

$$\begin{aligned} A^{-1\alpha}{}_{\beta} &= f_{-1}\,{}_{-1}P^{\alpha}{}_{\beta} + f_0\,{}_0P^{\alpha}_{\beta} \\ &\quad + f_{+1}\,{}_{+1}P^{\alpha}{}_{\beta} + f_2\,{}_2P^{\alpha}_{\beta} \end{aligned} \tag{B.15}$$

with

$$f_{M'} := 1/A_{M'}\,, \quad M' = -1, 0, +1, 2\,.$$

Using the tensors $2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta}$, $2\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta}$ (B.10) one gets

$$\begin{aligned} A^{-1\alpha}{}_{\beta} &= f_0\,{}_0P^{\alpha}_{\beta} + f_+\,2\,\mathrm{Re}\,{}_1P^{\alpha}{}_{\beta} \\ &\quad + i\,f_-\,2\,\mathrm{Im}\,{}_1P^{\alpha}{}_{\beta} + f_2\,{}_2P^{\alpha}_{\beta} \end{aligned} \tag{B.16}$$

with

$$f_{\pm} = \tfrac{1}{2}\,(f_{+1} \pm f_{-1}) = \pm\,A_{\pm}/(A_+^2 - A_-^2)\,.$$

[1] S. W. Lee and Y. T. Lo, J. Appl. Phys. **38**, 870 (1966).
[2] C. T. Tai, Radio Sci. **69 D**, 401 (1965).
[3] H. Heintzmann, W. Kundt, and J. P. Lasota, Phys. Rev. **A 12**, 204 (1975).
[4] H. Heintzmann and E. Schrüfer, Phys. Lett. **60 A**, 79 (1977).
[5] H. Hebenstreit and K. Suchy, Kleinheubacher Ber. **21**, 135 (1978).
[6] E. Schmutzer, Relativistische Physik, Akad. Verlagsges. Leipzig 1968, Chap. 4.
[7] H. Hebenstreit, Constitutive Relations for Moving Plasmas, Z. Naturforsch. **34**, 147 (1979).
[8] H. van Erkelens and W. A. van Leeuwen, Physica **89 a**, 225 (1977), Eqs. (3.4), (3.8).
[9] D. M. Fradkin, J. Phys. **A 11**, 1069 (1978), Eqs. (2,5), (2.6).
[10] U. Sturhann and K. Suchy, Z. Naturforsch. **31 a**, 1514 (1976), Eqs. (9), (16), (17).
[11] H. Derfler and R. A. O'Sullivan, Phys. Rev. **A 8**, 2645 (1973).
[12] G. Marx, Acta Phys. Hung. **3**, 75 (1954).